ANALYSIS I FINAL EXAMINATION

Total marks: 50
Time: 3 hours
Attempt all questions. If you use a result proved in class, please quote it clearly and completely.

(1) Let $a, b \in \mathbb{R}$ with $a < b$, let $f : (a, b) \to \mathbb{R}$ be a continuous function. Discuss the validity of the following statement: f is uniformly continuous, if and only if, it is bounded. Justify your answer. (8 marks)

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, satisfying $lim_{x\to\infty} f = 0$ and $lim_{x\to-\infty} f = 0$. Prove that f is bounded on $\mathbb{R}$, and f attains a maximum or minimum on $\mathbb{R}$. Give examples to show that f may attain a maximum but not a minimum, and viceversa. (7 marks)

(3) Does there exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ which takes every real value exactly twice? Justify your answer. (7 marks)

(4) Let $f : (0, \infty) \to \mathbb{R}$ be a function which is differentiable on $(0, \infty)$. Suppose $lim_{x\to\infty} f(x)$ and $lim_{x\to\infty} f'(x)$ both exist (as real numbers). Prove that $lim_{x\to\infty} f'(x) = 0$. (7 marks)

(5) Let $f : I \to \mathbb{R}$ be a function on an interval I having the intermediate value property (that is, for any two points $a, b \in I$, and any real number k between $f(a)$ and $f(b)$, there exists some c between a and b such that $f(c) = k$). Is f necessarily continuous? Justify your answer. (7 marks)

(6) Let f be continuous on $[a, b]$ and assume the second derivative f'' exists on (a, b). Suppose that the graph of f and the line segment joining the points $(a, f(a))$ and $(b, f(b))$ intersect at a point $(c, f(c))$ where $a < c < b$. Show that there exits a point $d \in (a, b)$ such that $f''(d) = 0$. (7 marks)

(7) Does the limit $lim_{x\to 0+} (sin(x))^x$ exist? If yes, what is it? (7 marks)

Date: October 29, 2015.